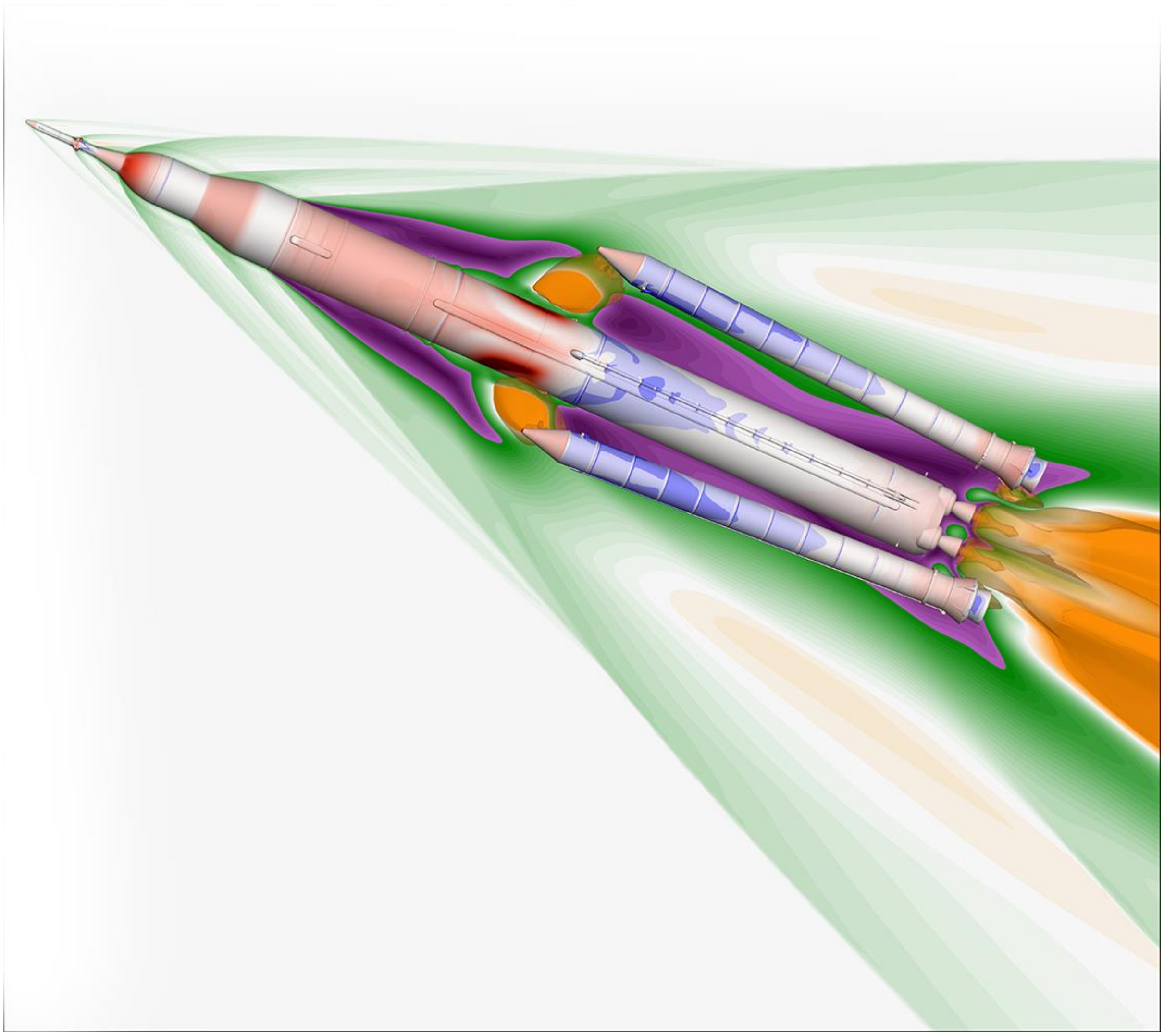


FLUIDS 4

CFD 2 – THE GOVERNING EQUATIONS OF FLUID MECHANICS

This section looks at the partial differential equations which govern fluid mechanics.

FLUIDS 4

CFD 2 – THE GOVERNING EQUATIONS OF FLUID MECHANICS

OVERVIEW

In this section you will learn about:

- The physical basis behind the governing equations.
- Why the differential versions of the equations are important.
- The differential version of the continuity equation.
- The Navier-Stokes equaton(s).
- The differential energy equation.
- What is missing and can be added to the basic equations.
- What simplifications can be made for easier solution.
- How the equations relate to CFD.

ASSUMED KNOWLEDGE

In this subject it is assumed that you already have knowledge about the following topics:

- *A good knowledge of fluid parameters*
- *Incompressible flow including Bernoulli, Continuity, Reynold's Number and Thrust equations*
- *Compressible flows and shockwaves*
- *The basic ideas behind CFD and some practice of its use*

These notes should be read after the CFD 1 notes on numerical methods.

Cover image: "SLS Block 1B booster separation flowfield" by NASA. Public domain, NASA terms and conditions (see page 13)

OBJECTIVE

The overall objective of this section is to learn about the differential form of the continuity, momentum and energy equations in fluid mechanics and how these relate to CFD.

TOPIC 1 – THE CONSERVATION LAWS

The equations presented here are known as the “governing equations” of fluid dynamics. They govern in the sense that they are the most general equations, apply to all normal fluids, and all the other equations we’ve covered in the degree (the continuity equation, Bernoulli’s equation, the energy equation, etc) can be derived from them. However, they are too complex to use directly in their most general form – they are simultaneous partial differential equations – but they are not too complex for modern computers to solve using the numerical methods already discussed – and this is the basis of CFD.

The equations, like all the others in fluids are derived from three principles:

- The conservation of mass – All fluid flowing into a closed system must also flow out of it.
- The conservation of momentum – Momentum (like energy) is conserved, this is often given form in Newton’s second law $F = ma$, meaning that any acceleration in a fluid is the result of the sum of all the forces acting on it (for example from pressure) and its mass.
- The conservation of energy – The energy flowing out of a closed system is the same as the sum of the all the energy flowing in,

The equation which results from mass conservation is called the “differential continuity equation”; those resulting from momentum conservation is called the “Navier-Stokes equations”; finally, we also have the “differential energy equation”.

TOPIC 2 – PRINCIPLES OF DERIVATION

We will not derive all the equations in these notes – you can work your way through the derivations in textbooks or on Youtube videos as homework (this is not directly examinable, but you should have a basic idea about where the formulae come from). Instead we will present the equations in a “common sense” way in the sections below. However, let’s just spend a few moments considering how the derivations work, so that you can follow these in textbooks and also gain insight into the form of the equations.

Because the equations are differential, we are concerned with instantaneous gradients and tiny volumes. Most of the derivations start by considering an infinitely small control volume as shown in figure 1 – off-course this is just a theoretical construct.

The sum of all the fluid mass flowing through all the faces of the volume must equal zero (that flowing in, must equal that flowing out). Momentum *in* must equal that *out* and similarly with energy.

Infinitesimal volume unit

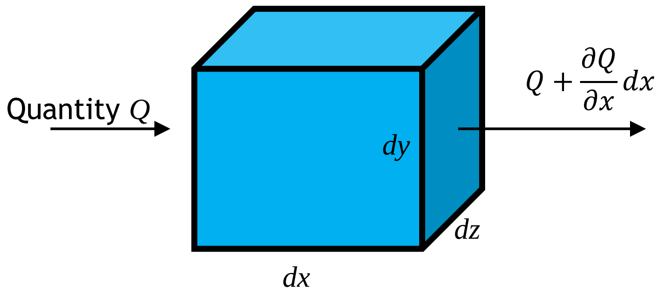


Figure 1, an infinitely small volume unit of the type usually used to derive governing equations.

TOPIC 3 – THE ACTUAL EQUATIONS

i) The continuity equation

The continuity equation (which we know as $\rho v A = constant$) can be written as a differential equation:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \hat{v}) = 0$$

or, to expand the *grad* (∇) symbol out:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

This looks horrendously complex, but actually it's just saying exactly the same thing as the simple version:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \hat{v}) = 0$$

A change in density (over time) + a change in mass (due to stuff flowing in or out through each surface) = 0

It can also be written as an integral equation:

$$\frac{\partial}{\partial t} \oiiint_V \rho dV + \oiint_S \rho \hat{v} \bullet d\hat{s} = 0$$

Change of density In the volume (over time) + Mass flowing through each surface = 0

So, why write the equations in such a complex form? One reason if that these equations are more accurate and specific than the general simple equation – they break down all the different contributions to the flow from changes of density with time and changes of speed in different directions and make write them down as instantaneous (not average) values.

Another reason is that it's possible to write the fluid flow pattern as a complete vector field – for example a velocity field.

$$v_{total} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

where v_x, v_y, and v_z are the velocity components in these three directions. Similarly, field equations can be written for other variables like pressure, temperature and density. Such a mathematical construction can capture all the subtleties of the flow and the differential equations can then be applied directly to these fields.

The differential forms are the basis of the FDM approach and the integral versions of the FVM. We'll concentrate on the differential versions here since we have discussed FDM in more detail in the previous set of notes.

ii) The momentum (Navier-Stokes) equations

Let's look at the biggest set of governing equations now. We'll start by developing an expression for the acceleration of a particle of fluid in a flow. Assuming that the flow is constant:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \qquad \text{(chain-rule trick)}$$

If the flow does vary with time, we need to add a further term:

$$a = \frac{dv}{dx}v + \frac{dv}{dt}$$

This is just for a one-dimensional flow – but it can be written in three dimensions:

Acceleration in the x, y and z directions

$$a_x = \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z + \frac{\partial v_x}{\partial t}$$

$$a_y = \frac{\partial v_y}{\partial x}v_x + \frac{\partial v_y}{\partial y}v_y + \frac{\partial v_y}{\partial z}v_z + \frac{\partial v_y}{\partial t}$$

$$a_z = \frac{\partial v_z}{\partial x}v_x + \frac{\partial v_z}{\partial y}v_y + \frac{\partial v_z}{\partial z}v_z + \frac{\partial v_z}{\partial t}$$

Now that we have an expression for acceleration, let's write down Newton's Second Law:

$$m\,a = F$$

As you know, in fluids, instead of the mass m, we usually use the density ρ. But what about the forces acting? Well, actually, there's normally three: 1) Gravity mg (or in a fluid ρg). 2) Pressure force $\frac{dp}{dx}$ (actually, the pressure gradient - nothing would move if surrounded by just a constant pressure). Finally, 3) the force generated by the fluid's viscosity $\mu \frac{d^2v}{dx^2}$ (this one is slightly more complex to derive, so I won't bother here, but you can look it up in a textbook).

We can use these ideas to write down the most important equations in advanced fluid mechanics (based on $F = m\,a$) – the *Navier-Stokes equations*.

$$\rho\left(\frac{\partial v_x}{\partial t}+\frac{\partial v_x}{\partial x}v_x+\frac{\partial v_x}{\partial y}v_y+\frac{\partial v_x}{\partial z}v_z\right)=\rho g_x-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v_y}{\partial t}+\frac{\partial v_y}{\partial x}v_x+\frac{\partial v_y}{\partial y}v_y+\frac{\partial v_y}{\partial z}v_z\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\underbrace{\rho}_{m}\underbrace{\left(\frac{\partial v_z}{\partial t}+\frac{\partial v_z}{\partial x}v_x+\frac{\partial v_z}{\partial y}v_y+\frac{\partial v_z}{\partial z}v_z\right)}_{a_z}=\underbrace{\rho g_z}_{F_{gravity\,(z)}}\underbrace{-\frac{\partial p}{\partial z}}_{F_{pressure\,(z)}}+\underbrace{\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)}_{F_{viscosity\,(z)}}$$

m a$_z$ = F$_{gravity\ (z)}$ + F$_{pressure\ (z)}$ + F$_{viscosity\ (z)}$

Assuming, of course, that ρ and μ are uniform (in other words the flow is incompressible and homogeneously Newtonian). If ρ wasn't constant (if the flow were compressible) then each differential term in the acceleration would have a sperate ρ value, for example:

$$\frac{\partial \rho v_x}{\partial x} \text{ instead of } \frac{\partial v_x}{\partial x}$$

Simple isn't it!

You may also see the following terminology applied to the equations (just taking the last equation as an example):

$$\underbrace{\rho\left(\frac{\partial v_z}{\partial t}+\frac{\partial v_z}{\partial x}v_x+\frac{\partial v_z}{\partial y}v_y+\frac{\partial v_z}{\partial z}v_z\right)}_{\text{Convective (advective) terms}}=\underbrace{\rho g_z-\frac{\partial p}{\partial z}}_{\text{Source and rate of increase terms}}+\underbrace{\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)}_{\text{Diffusive terms}}$$

The term *convective* in this context simply refers to movement or ordered motion of the flow (in other words, these terms describe the flow motion). The *source* and increase terms describe the forces on the flow and how they change (and we could add any other forces in here as well). Finally, *diffusive* refers to how the movement (kinetic energy) of the flow diffuses into the fluid due to its viscosity (you might call this a loss, but it's really just an increase in entropy caused by friction).

You may also see the spatial derivatives represented by the del (∇) operator – for example, using vector notation, we could write the whole equation system as:

$$\rho\frac{\partial \boldsymbol{v_{total}}}{\partial t} + \rho[\boldsymbol{v_{total}} \cdot \nabla \boldsymbol{v_{total}}] = \nabla \boldsymbol{P} - \rho g + \mu \nabla^2 \boldsymbol{v_{total}}$$

Apart from the in very simplest cases, like many other PDEs, the Navier-Stokes Equations can't be solved analytically – so they have to be solved *numerically* using computers - and one of the techniques we studied in the first CFD section.

iii) The energy equation

This equation describes heat added to the flow or heat exchanged within it (by friction) – so it is necessary to describe compressible flows and those with a heat source (like combustion or a heat exchanger). I won't go into the details here but just quote the equation (you can see intuitively what's happening when if you compare the form of this equation with the Navier-Stokes equation – many equations of this type have this structure):

$$\frac{\partial T}{\partial t} + \frac{\partial (v_x T)}{\partial x} + \frac{\partial v_y (T)}{\partial y} + \frac{\partial (v_z T)}{\partial z} = \frac{\partial}{\partial x}\left[\frac{v_x}{Pr} + \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{v_y}{Pr} + \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[\frac{v_z}{Pr} + \frac{\partial T}{\partial z}\right] + S$$

Change of temperature (in time and space) (convective terms) | Diffusive (dissipation) terms | Source term

TOPIC 4 – WHAT'S MISSING FROM THE BASIC EQUATIONS?

To these basic equations, other terms can be added, either to increase accuracy or to deal with special circumstances. For example, in standard flow scenarios these equations don't describe turbulence well, and a set of formulae to account for this is often added. Also, there are missing forces which are normally considered negligible, but would need to be accounted for in some circumstances – a good example of this would be surface-tension in the case of microfluidics. Other common special circumstances which would need addition or modification might be (among others):

- Reacting flows (flows with chemical reactions happening in them).
- Multiphase flows.
- Multi-viscosity flows.
- Non-Newtonian flows
- Low density flows.

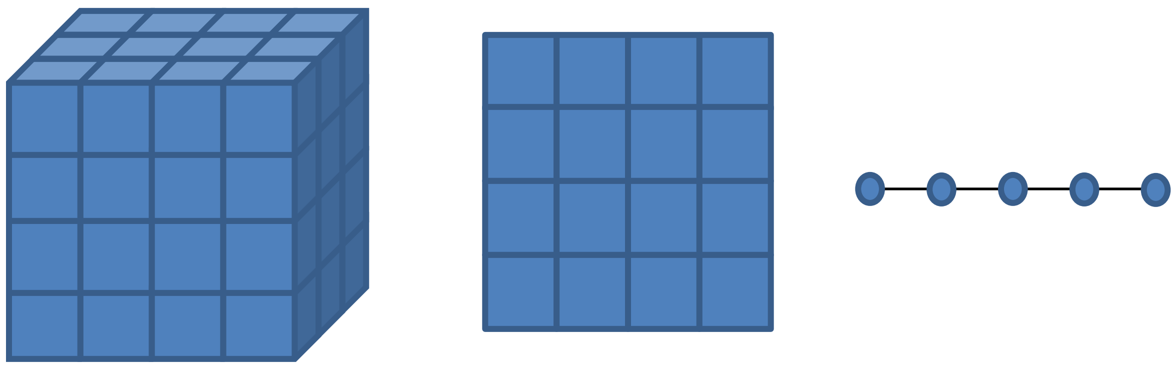
And so on.

TOPIC 5 – SIMPLIFICATION OF THE EQUATONS

Very often we don't need the full set of equations – or we can simplify them. This may either be because the flow is simple and we're neglecting some terms (making them zero). For example:

- Invisid flows (suitable models when viscosity is small or isn't important).
- Incompressible flows (slow gasses or liquids).
- Steady state flows (without transient effects).

Or when we don't need as many spatial dimensions – for example where a 2D or even 1D model is sufficient as shown in figure 2, overleaf.



Full 3D grid 2D grid 1D line

Figure 2, reducing grid dimensionality.

As an example, consider the continuity equation in two dimensions (x and y) for incompressible flow:

ρ is constant

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{=\,0} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

The first term is zero because density does not vary with time (the flow is incompressible), the z disappears (because we are only interested in x and y) and the other density terms are constant – so we get:

$$\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right] = 0$$

Divide through by the density:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Or inviscid 3D flow in which the effect of gravity can be neglected (I'll write this as the vector equation to save space):

$$\rho \frac{\partial \boldsymbol{v_{total}}}{\partial t} + \rho[\boldsymbol{v_{total}} \cdot \nabla \boldsymbol{v_{total}}] = \nabla \boldsymbol{P}$$

The inviscid version of the Navier-Stokes equations are sometimes called the *Euler Equations*.

In some simple applications (1D, incompressible flows), only the Navier-Stokes equations are necessary to simulate the flow as in the task examples below.

TASK 1

Write down the following equations and answer the other questions:

a) *From the Navier-Stokes equations, write down a 1D equation in the x direction using convection only (no source or diffusive terms).*

   This equation is known as the inviscid Burger's equation – explain what sort of flow you think it could represent?

b) *Similarly, a 1D equation for pure diffusion, but still including the variation of speed with time (but not space).*

   This is known as the Fourier equation and is important as, apart from fluid flow, it can also represent many diffusive processes – for example heat conduction in 1D time and space or the diffusion of one substance into another.

c) *The Fourier equation above can be discretized and is stable enough to be used in explicit mode, write down the difference version of the equation using a forward difference in time and a central difference in space. Rearrange this to give the next value of velocity (v_{t+1}).*

Continued overleaf

TASK 1 (continued)

You may need to give this some thought - to do this problem you will need a variable for velocity in space (v_i) *and one in time* (v_t) – or if you prefer velocity v at point i and time t perhaps labelled $v_{i,t}$.

Draw a stencil to represent this operation. If your lecturer hasn't explained what a stencil is then look it up:
[*https://en.wikipedia.org/wiki/Stencil_(numerical_analysis)*](https://en.wikipedia.org/wiki/Stencil_(numerical_analysis))

d) *Write down the equations of 2D flow with convection only (there will be two simultaneous equations).*

Note that in this case, because the flow is two dimensional, the continuity equation would also be necessary in a simulation.

e) *Write down the 1D viscid Burger's equation (just convection and diffusion). What type of flow would this represent?*

SUMMARY

- The governing equations of fluid mechanics are based on three conservation laws: Mass, Momentum and Energy.
- These are expressed as differential or integral equations.
- The equations can also be expressed in several other different ways.
- Other forces or effects sometimes also need to be added to the basic equations.
- The differential form is the basis of the FDM CFD and the integral form of the FVM CFD.
- The momentum equations are known as the Navier-Stokes equations (or in their inviscid form the Euler Equations).
- The equations may be simplified by neglecting unimportant terms or reducing the number of dimensions calculated. In some simple cases the continuity or energy equations are not required.
- The basic flow terms of the equations are sometimes referred to as “convective” or “advective” and the viscosity or loss effects as “diffusive” (or “dispersive”).
- Simpler forms of the equations have a number of other names including “the Burger equation” and “the Fourier equation”.
- The equations are discretized for CFD using one of the methods described in the first set of notes.

Note that this is a complex area with many subtle details – if you want to know the full story, study the references below.

REFERENCES, OTHER MATERIAL AND BIBLIOGRAPHY

If you are interested in delving into CFD in more detail, there is an excellent but slightly old-fashioned (master's level) course available on Youtube by Lorena Barba of Boston University:

https://www.youtube.com/playlist?list=PL30F4C5ABCE62CB61

A derivation of the governing equations can be found in the books below or:

https://www.youtube.com/watch?v=35unQgSaT88

https://www.youtube.com/watch?v=NjoMoH51UZc

If you find these too difficult to follow, there are many others on Youtube.

Two particularly recommended CFD books are:

John D Anderson, Computational Fluid Dynamics: The basics with applications, McGraw-Hill, 1995 (several editions).

Jiyuan Tu, Guan-Heng Yeoh and Chaoqun Liu, Computational Fluid Dynamics: A practical approach, Butterworth-Heinemann (Elsevier), 2013 (2nd ed).

There are Wikipedia pages are excellent "jumping off points" for further study:

https://en.wikipedia.org/wiki/Computational_fluid_dynamics

https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

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